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ALLOMETRIC GROWTH IN URBAN AND REGIONAL SOCIAL-ECONOMIC SYSTEMS

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Introduction

A process common to all aspects of urban and regional social-economic systems is growth, a generic construct that encompasses both increase and decline. Growth is a highly complex phenomenon that may manifest itself in a variety of specific types. Nevertheless, all forms of growth do have certain common characteristics. Boulding [5] has attempted to make these commonalities explicit in his three-fold classification of growth phenomena: (1) simple growth, (2) population growth, and (3) structural growth. This classification may be generalized further by distinguishing between absolute growth and relative growth.

Both simple growth and population growth are forms of absolute growth, as they deal with the accretion or depletion of some quantity over time. The primary goal of the analysis of absolute growth is that of finding a law of growth that will express the size of the growing variable as a function of time. Relative or structural growth, in contrast, involves time-independent changes in the relationships (generally those of a spatial nature but, by extension, sometimes including aspatial ones) of the elements of a system. Stated another way, relative growth involves differential morphological development. Since the three classes of growth identified by Boulding are not mutually exclusive, relative growth necessarily involves accretion or depletion over time. Time-dependent magnitude is not a primary concern, however, as it is not a basis for understanding the long-run structural evolution of a system.

A major portion of research involving demographic trends and economic development has been concerned with absolute growth. The more fundamental aspect of the growth of a region or an urban area, however, may be its relative growth, the time-independent structural changes in the relationships of its elements, both spatial and non-spatial. It is the goal of this paper to explore some of the implications for regional analysis of the theory of allometry, a tool for the investigation of differential morphological development. Major emphasis is placed upon the expansion of the conceptual basis for the application (and, thus, for the utility) of allometry in regional studies. The examples employed are simple ones and are intended to illustrate the broad applicability of allometry, rather than to provide definitive analyses of the data.

Definition

Broadly defined, allometry refers to the study of size and its consequences, and relates the differences in proportions of one component of a system to changes in either the absolute magnitude of the system or a second component of the system. [1]

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[1] The most comprehensive discussion of the theory of allometry is provided by Gould [14].

The principle of allometry dates at least as far back as Galileo (ca. 1638), but received its first formal interpretation by biologists in the mid-nineteenth century [3, p. 164]. Modern interest in allometry in biology and other fields is a direct consequence of Sir Julian Huxley's work in the 1930s. The literature of biology and of other fields [4; 22] contains numerous cases of documentation of the manner in which both organic and inorganic systems grow so as to yield a change in proportions.

The change in proportions or in the shape of a system is required by elementary geometry, specifically by the area-volume relationship. Note that this statement is to a large extent a "short-cut" rationale and should not be taken to imply that size increase is the efficient cause of shape alteration [14, p. 588]. If geometrical similarity is maintained with size increase, any series of objects will exhibit continually decreasing ratios of surface area to volume. Area varies as the second power of length, and volume varies as the third power. Constant area-to-volume ratios, an adaptive necessity for many organic relationships, can only be maintained by altering shape.

Consider the growth of a hypothetical cube-shaped organism in which one unit of surface area is required to furnish the necessary food, light, and air to support each unit of volume. If the proportions of the system remain constant, an increase in system size would cause the volume to exceed surface area at some point, since volume increases as the third power of length, and area as the second power; at this point the growth of the system must stop, as the surface is no longer able to act as an adequate interface between the organism and the environment. As illustrated in Table 1, area exceeds volume until the side length of such a cube is equal to six. In order for the organism to grow to a larger size, area must be allowed to increase more rapidly than volume. This requires that the basic dimensional relations and, thus, the geometry of the system be altered. If it were to continue its growth, the organism could no longer retain its cubic shape but would necessarily become convoluted or "bumpy".

Table 1

HYPOTHETICAL ORGANISM IN CUBIC FORM

Side Length	Area	Volume	Circumference
1	6	1	4
2	24	8	8
3	54	27	12
4	96	64	16
5	150	125	20
6	216	216	24
7	294	343	28

The basic allometric relationship is described by a power function which has come to be known as the "allometric equation":

$$y = bx^a$$

or its equivalent form,

$$\log y = \log b + a \log x$$

where x represents the size of the entire system, or a portion of that system that is being used as a frame of reference; y is the size of a particular element of that system; b is the y intercept when log x equals zero; and a is the exponent, the coefficient that relates the amount of change in log y per unit change in log x.[2]

Some authors, including D'Arcy Thompson [29], argue that the use of the logarithmic power function is inappropriate and that many of the trends described by it are equally well rendered by linear regressions. The power function, however, has been used almost exclusively, as it combines an adequate statistical fit with simplicity and interpretability [14, p. 596]. Further, since growth is multiplicative in the general sense that what is produced by growth is itself normally capable of growing, it is reasonable to compare growth on a logarithmic scale where addition of units represents a multiplicative effect [15, p. 281].

On the basis of the value assumed by the exponent, three categories of allometry can be distinguished. The first category is that of positive allometry. This means that y has a differentially large increase relative to x; the exponent a, which represents the ratio (log y/log x), is greater than one. The second is negative allometry. Here y decreases relative to the increase in the magnitude of x; the exponent is less than one. Finally, when x and y maintain a one-to-one correspondence throughout their increase, growth is said to be isometric; the exponent is equal to one. These definitions of positive and negative allometry and isometry are valid, however, only when the x and y parameters have the same dimensionality. When the dimensions of x and y are not equivalent, isometry is indicated by the ratio of the dimensionality of y to the dimensionality of x. For example, when y is an area $[L^2]$ and x is a volume $[L^3]$, as in the case of the hypothetical cubic organism, an exponent of 2/3 is indicative of isometry; a value greater than this denotes positive allometry; a lower value denotes negative allometry.

In the allometric power function it is both the exponent a and the intercept b which relate the growth of the system component y to the growth of the system, or the component, x. The exponent represents the change in the value of log y per unit change in log x, indicating the general nature of the co-relationship. The specific value of y for a given value of x, however, is also a function of the value of the intercept. For this reason, the role of a criterion for the "intensity of differential increase" is generally ascribed to the intercept [14].

Three Related Interpretations

The application of the principle of allometry to the social sciences is by no means novel. Pareto's [22] law of the distribution of income, the articulation of the "rule" of rank-size regularities in the population of cities by Auerback [1] and Zipf [39], and Zipf's [38] law of the distri-

[2] The power function equation may be derived in this manner:

$$F(dy, y, dt) = ay$$
$$F(dx, x, dt) = ax$$

and since the growth functions of y and x can be said to cancel algebraically,

$$\frac{dy}{ydt} \; \frac{dx}{xdt} = a$$

$$\int \frac{dy}{ydt} = a \int \frac{dx}{xdt}$$

and eliminating dt from both sides,

$$\log_e y = a \log_e x + \log_e b$$
$$y = bx^a.$$

bution of word frequencies in languages, are implicit aspatial formulations of the principle. A number of examples of the application of allometry to the spatial characteristics of human and physical phenomena may be found in the geographic literature, commencing with the work of Stewart and Warntz [26] two decades ago.[3] The vast majority of these studies, however, have taken very limited views of allometry, focusing on the existence of a constant ratio relating the increase of the x and y parameters, as indicated by the exponent. Generally neglected have been the change of shape of a system necessitated by increasing magnitude, as exemplified by the simple model of the cube-shaped organism, and, to an even greater extent, the implicit expression of competition in the allometric relationship.

The Constant Ratio

The most common use of allometry in the social sciences has involved fitting a power function to a set of data describing the growth of a system. In cases where the power function represents a good fit to the data, the exponent may be interpreted as a descriptive measure of the manner in which the relations between system elements change with increasing size. This may be illustrated by Bunge's [7; 8] interest in the relative proportions of the components of a city system. He has suggested that the "slum" is a functional part of a city and that by increasing the size of the city, the size of the slum will necessarily increase; the specific nature of the increase is indicated by the exponent. A further example involves the relationship between the areal extent of a spatial unit and its population size. Stewart and Warntz [26] have found that a constant ratio exists between the areas and populations of certain city systems. The exponent of the power function relating these two parameters may reflect an optimal spatial expression of growth.

Size-Correlated Shape Change

Several geographers who have utilized the allometric principle have incorrectly applied the term "allometry" to any relationship that may be fitted by a power function. Nordbeck [19; 20], for example, equates the term "law of allometric growth" to "power function" and then proceeds to use the power function only for relations exhibiting the special case of isometry, where there is no change of shape with size increase. Since allometry means size-correlated change of shape (this is implicit in its etymology), these types of studies reflect a confusion over the proper use of the principle.

The term "shape change" is a general one that includes alteration both of the external form of an object, such as the convolution of the surface of the hypothetical cubic organism above, and of the internal relations of the components of a system. When growth exhibits either positive or negative allometry, the shape of a system cannot remain unchanged beyond a certain size.

Nordbeck [20] discovered that an exponent of 0.66 relates the areal extent of built-up areas in Sweden to their population sizes. Since population (the x parameter) is regarded as a volume, having the dimensions of 3, and since area has the dimensions of 2, isometry is indicated by an exponent of $2/3 = 0.66$.[4] Therefore, no size-related shape change is exhibited in the Swedish built-up areas. Stewart and Warntz [26] found, on the other hand, that for a sample of European cities, area and population were related by an exponent of 0.75. Since 0.66 signifies isometry, 0.75 represents positive allometry and indicates that there is a change of shape associated with the variation in the size of these cities. The area-population relationship will receive further consideration below.

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[3]See, for example, Nordbeck [19; 20], Woldenberg [36; 37], Ray [23], Dutton [11; 13], and Strahler [27].

[4]Woldenberg [36] accepts this view but offers the alternative interpretation that population be regarded as an "area-using" variable and, therefore, of the same dimensions as area, namely 2. In this interpretation, isometry is indicated by an exponent of 1.0.

A final point concerning size-correlated shape change involves the notion of the limits to growth. Going back to the simple model of the cubic organism once again, recall that when the length of the side of the cube attained six units, the volume of the organism equalled its surface area. If the organism were to grow any larger, the volume would necessarily exceed surface area. Since this would surpass the ability of the organism to sustain life through interface with its environment, growth would have to cease when volume equalled area. As Boulding [5, p. 72] notes, growth creates form but form limits growth. The only manner in which the organism could continue to grow would be if the surface area increased disproportionately more than increase in volume; that is, if the exponent were lowered. Since

$$\text{volume} = b\ \text{area}^{3/2}$$

indicates isometry, an exponent with a value lower than 3/2 would enable this disproportionate increase. And, as we have seen, this decrease in the exponent results in convolutions and bumps in the basic cubic shape.

Competition

The final aspect of allometry is one which, to the best of my knowledge, has been articulated only by Bertalanffy [3]. Briefly, the allometric relationship may be viewed as an expression of competition within a given system, with each system component taking its share of the available resources of the total system according to its capacity, as expressed by the exponent. The exponent may thus be regarded as a "growth partition coefficient" [15, p. 49] that expresses the capacity of a component to seize its share of the resources. An exponent indicating positive allometry signifies that the component in question captures a proportionately larger share of the resources than either the total system or a second component. Conversely, an exponent indicating negative allometry signifies that a component captures a share proportionately less than the system or a second component.

The differential growth of various system components is, then, a consequence of the competition among these components for the resources available to the system from its environment. This is a direct extension of the theory of open systems, which concerns the continuous exchange of matter and energy between a system and its environment. In Biology, the role of competition both among the components of an organism and among the species that comprise an ecological system has long been recognized. Bertalanffy [3, p. 66] writes that every whole is based upon the competition of its elements and presupposes the struggle between parts, and Thom [28, p. 323] suggests that all morphogenesis may be attributed to this type of conflict. One consequence of a sustained advantage in favour of one component is that in the long run the components with the smaller capacities will be exterminated. In the extreme case of growth, carcinogenesis, one component captures so much of the avaliable resources that the total system not only grows proportionately less than the carcinoma but eventually declines in absolute size.

In biological instances of allometry, the resources of the environment that are the subject of competition generally include food, light, air, water and, not least, space. Indeed, Thom [28, p. 222] cites the competition for space as one of the most primitive (and therefore basic) forms of biological interaction, both among the internal components of an organism and among the components of an ecological system. In social-economic systems, the same basic form of competition may be identified. In fact, the arrangement of phenomena on the earth's surface may be generally conceptualized as the outcome of the competition for "equipped" space, space having particular characteristics.

One example that comes to mind involves Bunge's statements concerning the relationship of the size of the slum to the size of its "host" city. By extension, this also implies a relationship between the slum and the other components of the city; the upper class residential district, for example. Using several different criteria of size, we might expect to find, and to empirically fit the values of the two constants of, the following relationships between a city (C), its slum (S), and its upper class residential district (R):

$$\text{area } S = b \text{ area } C^{p} \qquad (1)$$

$$\text{population } S = d \text{ population } C^{q} \qquad (2)$$

$$\text{income } S = e \text{ income } C^{r} \qquad (3)$$

$$\text{area } S = f \text{ area } R^{t} \qquad (4)$$

$$\text{population } S = g \text{ population } R^{u} \qquad (5)$$

$$\text{income } S = h \text{ income } R^{v} \qquad (6)$$

These equations not only describe the relationships between the slum, the upper class ghetto, and the whole city, but, if the values of the exponents are known, also describe the outcome of the competition for the resources of the urban environment, space, people, and income. Since all of the equations are dimensionally balanced, an exponent greater than one indicates that the slum is capturing a proportionately larger amount of the resources; an exponent less than one indicates that it is capturing a proportionately smaller amount; an exponent of one indicates that it is capturing its "fair share". On the basis of a stereotyped image of the urban system, one might expect the exponent to exceed one in equations (1) and (2) and to be less than one in equations (3) and (6).

Allometry in Urban and Regional Systems

The following examples are intended to illustrate both the three facets of allometry and the wide range of phenomena to which the principle may be applied. .As these examples indicate, there are two distinct modes of analysis which may be employed: diachronic analysis, which traces the growth of a single system through time; and synchronic analysis, which examines a set of systems at a single point in time. The implicit assumption of synchronic analysis is that a set of individual systems of varying magnitude represents the stages of growth of a single system; small cities, for example, are assumed to have the same form as large cities and to represent their early stages of growth. Biologists conventionally make this assumption in studies of the growth of individual members of a particular species. As there may be qualitative as well as quantitative differences between small and large social-economic systems, the synchronic approach must be applied with caution.

Urbanization

The rapid population growth of many of the world's nations has been concentrated in that portion of the population which may be classed as

urban; the phenomena of metropolitanization and megalopolitanization have long been recognized over the globe. While this rapid urbanization has been occurring, however, there has been no more that limited success on the part of any society in either stimulating further urban growth or preventing it [13,p. 2]. This suggests that there may exist in social-economic systems some natural principle of growth that resists exogenous control.

Such a principle may be illustrated by the rate of Canadian urban growth, which has maintained a constant ratio to the rate of the nation's total population growth from 1901 to 1976 (Figure 1).[5] Although there is some difficulty associated with these data because of the changing definition of "urban", the fit is a good one ($r^2 = 0.999$) and the exponent (1.50) indicates that the urban population of Canada is increasing at a rate greater than that of the total population (positive allometry). Figure 1 also shows the growth of the urban fraction in the Maritime and Prairie regions, both of which exhibit positive allometry. The rate of growth of the urban population in the Maritimes is somewhat less than that of the national system, while that of the Prairies exceeds the national system. This is likely a reflection of the early urbanization of Eastern Canada and the more recent urbanization of the Prairies. Ideally, the analysis should commence with an earlier time period, but reliable data are not readily available.

The relationship between the total population and the urban fraction illustrated here may be conceptualized as competition between the urban and rural subsystems. The disproportionate increase of the urban portion of the population indicates the relative degree of attraction of urban places, and may be a manifestation of some functional imperative or design criteria within the national system. Further, the differential increase of the urban fraction has definite implications for the shape, the geometry, of the national system. The distribution and flow patterns of people, information, goods, and money in a highly urbanized system may be expected to be quite distinct from those in a rurally oriented society.

The empirical relationship presented here is consistent with the work of Stewart [25], later extended by Dutton [12], in which the exponent relating the urban fraction of the U.S. to its total population over the years 1790 to 1970 was determined to be 1.73. The allometric exponent, one might hazard, may have considerable significance as a parameter of growth which characterizes the nature of the urbanization process in differing economic, cultural, and technological mediums.

Land Use

It has been explicitly recognized by a number of researchers [2; 17] that land use proportions within a city change with the size of the city, as defined by either area or population. The methods of analysis employed by these researchers, however, have been somewhat unsophisticated. For example, size has not been treated as a continuous variable, making comparative studies difficult.

The allometric principle appears to have considerable utility for studies of land use proportions. The areas devoted to specific land uses within a given city might be expected to be functions of its total developed area, as opposed to the total area within the political subdivision. The exponents of the power functions relating land use areas to total developed area would then convey considerable information concerning the internal relations of urban land use.

An allometric analysis of urban land use may be illustrated using a data set compiled by the Ontario Department of Municipal Affairs [21].

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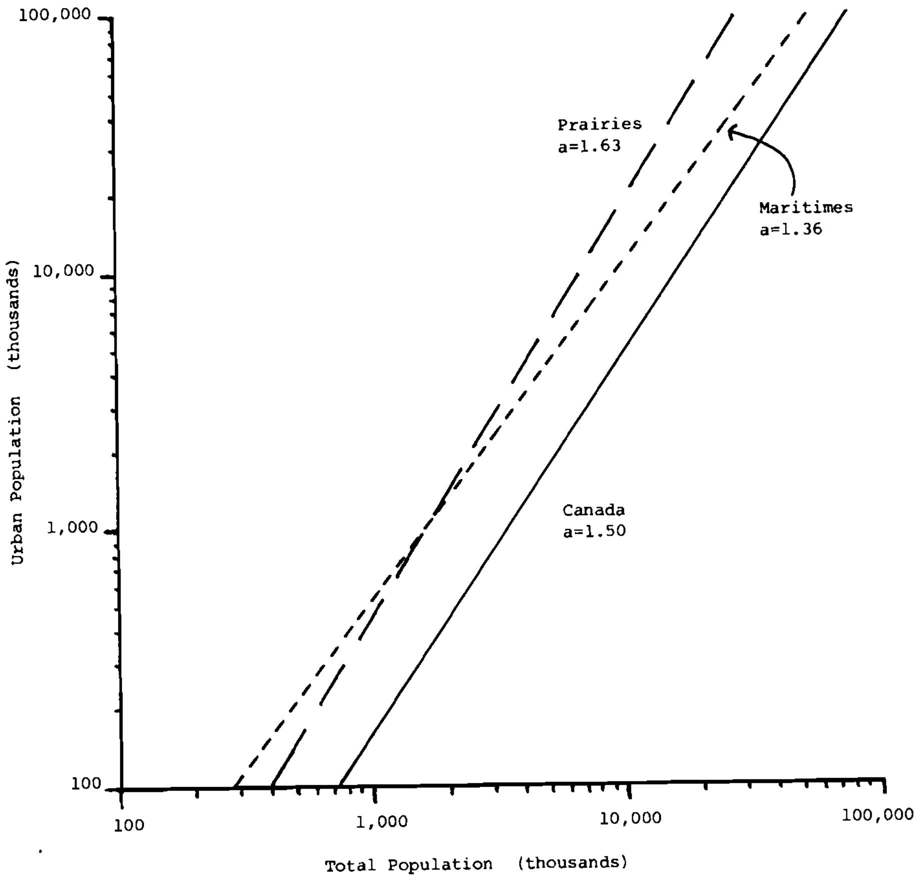
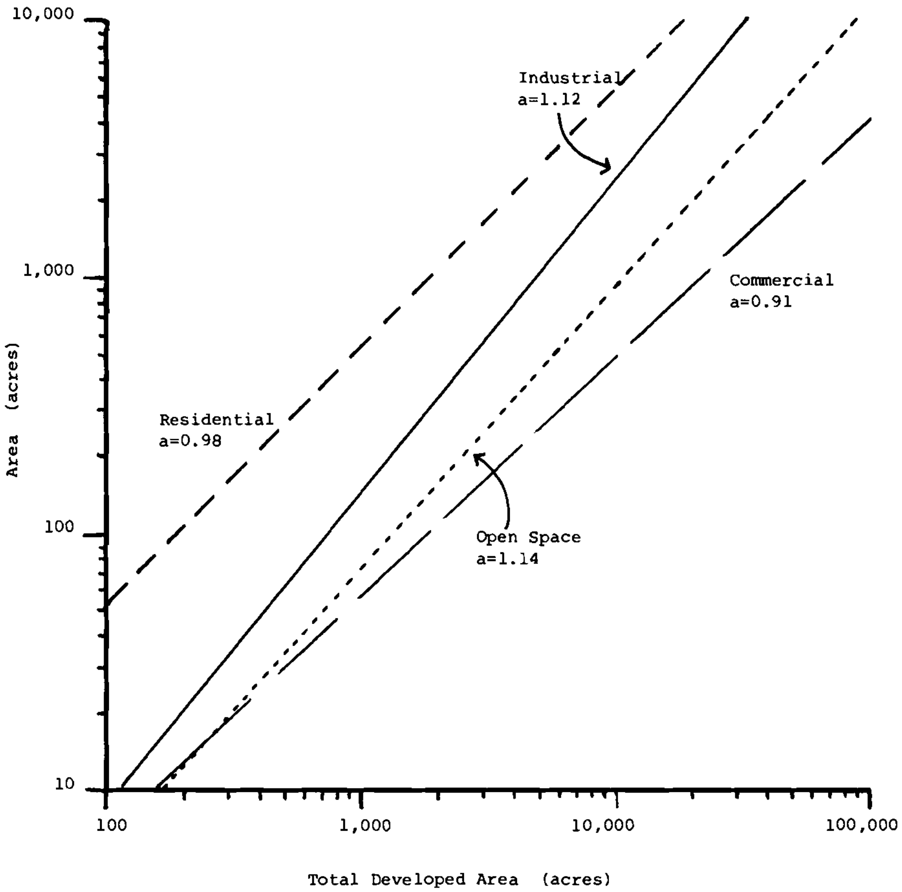
[5]Data source: Historical Statistics of Canada [32], Series A 15-19.



Canada	$P_u = 0.005\ P_t^{1.50}$	r= 0.99
Maritimes	$P_u = 0.434\ P_t^{1.36}$	r= 0.97
Prairies	$P_u = 0.006\ P_t^{1.63}$	r= 0.95

Figure 1

THE CANADIAN URBAN FRACTION, 1901-1976



Commercial	Area= $0.106\ TDA^{0.91}$	r= 0.96
Industrial	Area= $0.070\ TDA^{1.12}$	r= 0.95
Open Space	Area= $0.028\ TDA^{1.14}$	r= 0.91
Residential	Area= $1.521\ TDA^{0.98}$	r= 0.99

Figure 2

URBAN LAND USE, ONTARIO, 1966

This data set contains estimates for total developed area, population density, and acreages in residential, commercial, industrial and open space categories for 51 urban areas in Ontario.[6] These urban areas range in (1966) population size from Metropolitan Toronto (1,652,300) to Tottenham (700). Although obvious problems concerning criteria, classificatory procedures, and variability in data exist, the high level of aggregation employed renders the data sufficiently consistent for illustrative purposes.

Figure 2 displays the relationships between total developed area and four land use types for the set of cities. Industrial land and open space exhibit a slightly positive allometry; as the total developed area of a city increases, the area of land devoted to these categories increases by a greater proportion. Commercial and residential land, on the other hand, show slightly negative allometry. There is some question, however, as to whether these exponents differ significantly from 1.0. In other words, in all four cases growth may be isometric; specific land uses and total area may increase as a one-to-one correspondence. While one should not ascribe any great significance to the particular exponent values that have been arrived at here because of the nature of the data, it is evident that this type of analysis, combined with a reliable data set, may prove valuable in understanding the structure (shape change) and dynamics (competition) of a particular culture's urban system. The allometric principle may thus have strong and utilitarian implications for planners.

These results are consistent with Woldenberg's [36] analysis of U.S. data; commercial and residential land were found to exhibit negative allometry, while open space and industrial land showed positive allometry. In all cases, however, the exponents were close to 1.0. Woldenberg's investigation also considered the relationship between the various land use categories and the total population of cities, instead of total developed area. Similar results were obtained when total population was employed.

Population and Built-up Area

The initial, and perhaps most widely utilized, application of the allometric principle in geography concerns the relationship between the population and the built-up (total developed) area of settlements. A number of studies which have employed highly diverse data have concluded that the built-up area of a settlement may be related to its total population by a power function of the form:

$$\text{built-up area} = b \text{ total population}^{a}.$$

In general, the nature of the relationship is one of positive allometry; as a settlement grows, its areal extent increases more rapidly than its population.[7] In other words, population growth is spatially extensive, with the increasing numbers tending to become distributed on the periphery. Table 2 summarizes the exponents arrived at in several studies in the geographic literature. In addition, the exponent relating population and total developed area for the set of 51 Ontario settlements

[6] Ideally, the analysis of varying proportions of land use would involve diachronic data. There are, however, virtually no diachronic data available, and relatively few synchronic data sets.

[7] Recall that, due to the dimensions of the variables involved here, an exponent of 0.66 is indicative of isometry.

is shown. Note that Stewart and Warntz have established the value of the exponent using both synchronic and diachronic anaylsis.[8]

Table 2

EXPONENTS RELATING BUILT-UP AREA AND TOTAL POPULATION

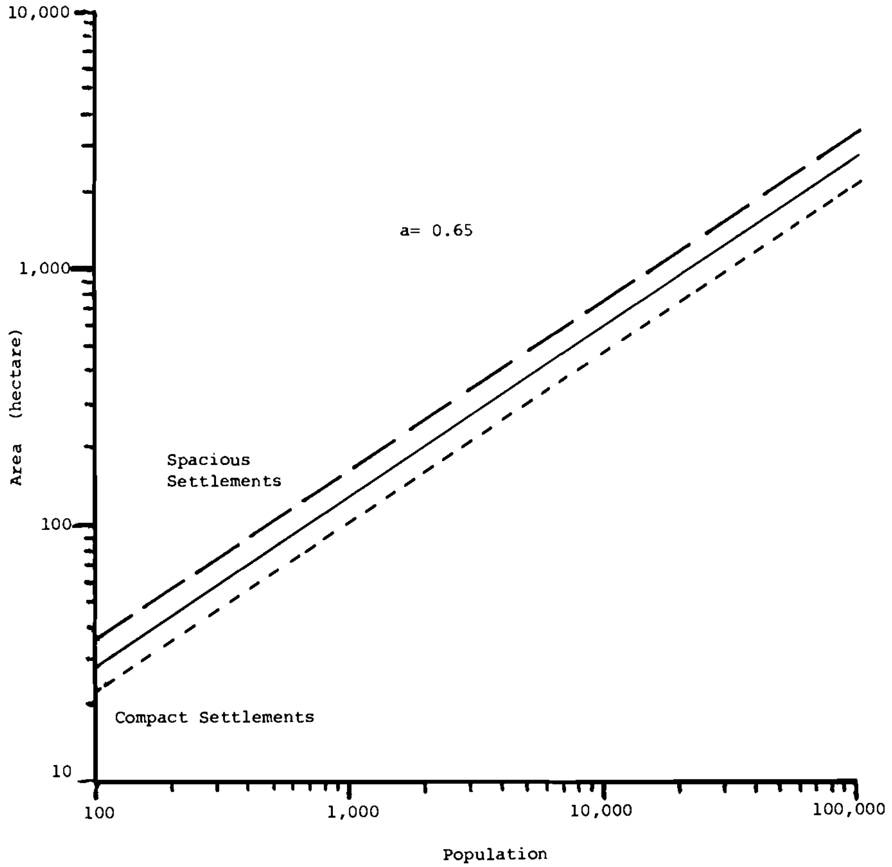
Author	Data Set	Exponent
Woldenberg [36]	89 U.S. cities, 1960	0.78
Nordbeck [19]	Swedish cities, 1960	0.66
	U.S. cities, 1950	0.86
	U.S. cities, 1960	0.88
	Densely inhabited districts, Japan, 1960	0.91
Stewart and Warntz [26]	U.S. and European cities, 1951	0.75
	U.S. and European cities, 1890-1951	0.75
Coffey	51 Ontario urban areas, 1966	0.87

The power function exponent has received considerable attention in these studies as it is quite readily interpretable in terms of the differential growth of area relative to population. The intercept of the power function, however, is quite possibly a more meaningful index of the area-population relationship. Where the exponents of two power functions are equivalent, the values of the intercepts indicate differences in the amount of area which is required to contain a given level of population. This is consistent with Gould's [14] identification of the role of the intercept in an allometric equation as a criterion for the intensity of differential increase.

Nordbeck's [20] investigation of the area-population relationship in Swedish settlements illustrates this interpretation of the intercept. Using the intercept as a "space standard" or index of compactness, he identifies three classes of settlement: those with compact built-up areas, having a relatively small area for a given population; those with mean built-up areas; and those with spacious built-up areas, having a relatively large area for a given population (Figure 3). The compact settlements have thier manifestation in railroad centres and older ports, whereas the spacious settlements are resorts.

In a similar manner, Tobler [31] uses the intercept of an equation relating the radii of circles proportional to the built-up areas of settlements to population as a measure of settlement packing. Examining the intercept value for the urban systems of a number of cultures, he notes that it possesses considerable usefulness in distinguishing between the various strategies employed by societies for the organization of spatial activity. Stewart and Warntz [26, p. 104] have suggested that the level of settlement packing, or land use intensity in a settlement, is related to the base population potential of the settlement; empirical

[8] Vining and Louw [33] have recently introduced into the literature a cautionary note on the power function relating urban area and population. Their results indicate that in some countries the function may not be stable over time; diachronic analysis should not be expected to replicate synchronic results.



After Nordbeck (20)

Figure 3

INDEX OF COMPACTNESS, SWEDISH URBAN AREAS, 1965

evidence indicates that at higher potentials the areas of cities tend to be constricted.

Income Density and Income Potential

A problem common to the preceding illustrations is the definition and measurement of the relative magnitude of the variables involved. The exponent values may vary considerably depending upon the criteria and the consistency with which urban population, total developed area, and the areas in specific land uses are defined. The problem is essentially one of drawing a boundary around a set of homogeneous elements. It involves partitioning continuous variables, such as population and area, into discrete classes for the purpose of measurement. One method of partially avoiding problems related to definition and measurement is to employ continuously distributed variables in the analysis.

Income potential is a continuous field quantity which is an aggregate index of accessibility to the income in a spatial system. Income density, although conventionally measured for discrete units, is a continuous quantity everywhere differentiable and generally exhibiting no discontinuity across a spatial system. These two parameters have been shown to be of considerable utility in describing the spatial structure and the spatial dynamics of a social economic system [34; 35].

Rosen [24] and Brody [6] have suggested that the allometric exponent behaves as a design criterion; the system changes geometrically in order to remain the same functionally. This notion is relevant within the present context. Since income potential generally declines in a symmetrical manner from the centre of a city, and since the relationship between income potential and income density is positive, locations toward the city centre will have densities higher than those locations toward the periphery. The specific income densities over a given potential distribution are a function of both the exponent and the intercept of the $D = bU^a$ power function. The level of density in a peripheral location relative to a central location, or, more precisely, at low potential relative to high potential, may be expressed solely on the basis of the value of the exponent, however, assuming that the intercept is constant.

Since income density has the dimensions of dollars per square mile, ($\$L^{-2}$), and income potential has the dimensions of dollars per mile, ($\$L^{-1}$), an exponent value greater than 2 indicates positive allometry. That is, an exponent greater than 2 signifies that any increase in income potential will be accompanied by an increase in income density such that the "shape" of the social-economic terrain changes in terms of its income density-income potential relations. Again, there is substantial theoretical basis for interpreting this change in geometry as permitting the system to maintain its functional integrity throughout. A higher level of income density is required in the centre of the city in order to maintain functional relationships commensurate with those achieved at lower densities on the periphery. One readily identified manifestation of this functional relationship involves the concept of returns to investment. Given the higher costs of developing and maintaining commercial and residential facilities in the central city, and given the low degree of physical mobility of inner city residents relative to suburban consumers, a higher density of income is necessary to guarantee returns on central city investments comparable to those that may be acquired at lower densities in the suburbs. At the scale of the U.S. national system, Warntz [34] has shown that the exponent has deviated little from a value of 3.0 from 1880 to 1959. Subsequent work by Warntz and this author has shown that the exponent maintains its value through 1970. Both Warntz and Tideman [30] provide theoretical justification for regarding this value as an optimal one which enables the system to maximize profit over space. This reinforces the notion of the exponent's role as a design criterion which indicates the requisite shape of the system.

Investigations at both metropolitan and national levels strongly suggest that the D - U. exponent further represents a generalized measure of intensity of land use. Within a metropolis there appears to be a direct relationship between the exponent and level of internal land utilization, manifest in high land rents and high densities. Further, evidence indicates that the exponent also reflects homogeneity of income distribution, the exponent being inversely related to degree of homogeneity. Theoretical support for this relationship is independently provided by Bunge's [7] adaptation of the von Thunen rent model, which demonstrates the association between high density of population and income and disparities in social well-being. At the national scale the exponent has similar significance, indicating level of urbanization or agglomeration and degree of homogeneity of income distribution. Thus, there is evidence that at either scale the exponent may be viewed as a relative measure of social-economic homogeneity or its converse, stratification.

If we compare two hypothetical cities with similar income potential profiles, it is evident that the larger the difference between the exponents of each, the greater the variation in the ranges of income density that each experiences. A wide range of income density values is generally indicative of stratification. It has been demonstrated elsewhere by the author that high income density is characterized by the crowding of large numbers of lower income persons onto a given tract of land [10]. It is precisely this condition that makes a "slum" one of the most profitable uses to which a parcel of central city land can be put.

Income density-income potential relationships in the Boston and Toronto metropolitan systems and the U.S. and Canadian national systems are summarized in Table 3. At both metropolitan and national scales the U.S. systems exhibit higher exponents than their Canadian counterparts. Note that both Canadian systems have intercepts higher than their American counterparts, indicating higher income densities at very low levels of potential. As potential values never decline to such levels within the boundaries of any of the systems, however, it is possible to say that the U.S. systems have higher densities at all levels of observed potential. The indication is that the U.S. systems possess a geometry (broadly defined in terms of both social and physical dimensions) that is more efficient in terms of Tideman's notion of the maximization of profit. Yet, as noted above, the cost of this efficiency is perhaps high in terms of equality of well-being.

Table 3

METROPOLITAN AND NATIONAL POWER FUNCTIONS: INCOME DENSITY (D) AND INCOME POTENTIAL (U)

	Equation	Exponent	Coefficient of correlation	Number of observations
Canada (1971)	$D = 7.48 \times 10^{-4}\ U^{2.14}$	2.14	0.813	238
Toronto (1971)	$D = 3.61 \times 10^{-14}\ U^{2.26}$	2.26	0.757	447
Boston (1970)	$D = 1.49 \times 10^{-17}\ U^{2.65}$	2.65	0.826	536
U.S. (1970)	$D = 2.90 \times 10^{-8}\ U^{3.03}$	3.03	0.735	3068

Conclusion

The preceding illustrations demonstrate that an allometric perspective may be of some utility in the analysis of regional and urban social-economic systems. The potential for the application of allometry lies both in the description of various aspects of growth and in attempts to achieve an understanding of the processes which are involved in regional development. In this latter regard, the role of the allometric exponent as a design criterion which places certain constraints upon system geometry, broadly defined in terms of physical space and N dimensional social space, may be of particular relevance. Strong evidence suggests that all systems, economic as well as biological, change geometrically in order to maintain their functional relations.

Three implications of the allometric principle have been explored. Whereas the constant ratio and the size-correlated shape change aspects are quite evident in the illustrations employed, the concept of competition is somewhat less readily interpretable. The growth of the urban fraction and differences in the proportions of land use types involve competition explicitly. The relationships between the population and area of settlements and between income potential and income density are, however, similar manifestations of competition. As in the former examples, the distribution of people and income over space is implied in these relationships, and competition may be conceptualized in terms of alternative strategies for the intensive or extensive utilization of available space.

Allometry involves a simple functional relationship between the sets of values assumed by the two system components under consideration. This simple functional relationship which describes the nature of growth in a system represents the optimal alternative under a given set of conditions. This has been demonstrated mathematically by Rosen [24, pp. 80-83]. Further, optimal forms are inextricably knit with conservation principles. Most systems that are in, or are approaching, an optimal state behave in a manner that conserves space, time and energy; in a social-economic system income is probably conserved also. This is not to suggest that all systems which are characterized by allometric growth are in an optimal state, but rather that there is a tendency for them to approach that state. The allometric relationship is also characteristic of open systems that are in dynamic equilibrium or a steady state. The steady state of an open system takes the place of and is equivalent to the equilibrium state of a closed system. Therefore, there is a direct connection between allometry and probability since, in a closed system, equilibrium, the state of maximum entropy, is the most probable state [35]. Allometry may thus be regarded as representative of a condition of maximum probability within the flow constraints of a given system.

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